Largest-Index Zonotope Bounds for Planar Projections of Semiorder Ideals

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Abstract

Let P be an n-element semiorder supplied with a utility representation, and assign an arbitrary rational vector in $\mathbb{Q}^2$ to each element. Every order ideal projects to the sum of its element vectors. We bound the number of strict vertices on the upper hull and on the full convex hull of these projected sums. Sorting by utility and partitioning nonempty ideals by their largest selected index expresses each part as a translated planar zonotope. If q_j is the number of earlier elements incomparable with index j, and $\operatorname{inc}(P)$ is the number of unordered incomparable pairs, the resulting bounds are

$$h(P,v) \le 1 + n + \operatorname{inc}(P), \qquad f(P,v) \le 1 + \sum_j \max\{1, 2q_j\} \le 1 + n + 2\operatorname{inc}(P).$$

For width at most w, with $r = \min\{n, w\}$, this gives $h(P,v) \le 1 + rn - \binom{r}{2}$ and an explicit full-polygon analogue. A mixed-sign chain construction gives $n+1$ vertices, showing the correct linear order in n for each fixed width. The result refines the small-width regime of the known $O(n \log n)$ semiorder bound. A standard-library exact verifier supplies finite falsification tests; the written argument, not the computation, is the proof.

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1 Introduction

For a finite partially ordered set, the maximum-weight closure problem asks for a maximum-weight order ideal. If every element weight depends affinely on a scalar parameter, the supported optima are encoded by the upper hull of a planar projection of the ideal incidence vectors. Eppstein [1] introduced this parametric closure formulation and proved, among other results, that the full projection polygon of an arbitrary n-element semiorder has $O(n \log n)$ vertices.

This note records an instance-sensitive refinement for represented semiorders. The key decomposition is elementary: after sorting by utility, the ideals with a fixed largest selected index form an affine Boolean cube, whose planar image convexifies to a zonotope. Its number of upper vertices is controlled by the number of free incomparable predecessors. Summing those local counts charges every incomparable pair exactly once.

The resulting theorem is useful in two regimes. First, it bounds a particular instance by $\operatorname{inc}(P)$ rather than by n alone. Second, for a semiorder of width r it gives $O(nr)$ vertices, which combines with Eppstein's theorem to give $O(n \min\{r, \log n\})$. It is therefore asymptotically smaller when $r = o(\log n)$, including every fixed-width class. It does not improve the known unrestricted semiorder bound when r is large, and it does not address the unrestricted parametric closure problem for general posets.

An earlier route counted all ideals. That route is subsumed by Wood's sharp clique bound for degenerate graphs [2] and is not presented as a contribution. The surviving argument counts supported planar vertices, not all ideals. A dated, bounded literature audit found no primary source stating the exact incomparable-pair or $O(nr)$ projection bounds. This search does not establish historical priority; no "first" or absolute novelty claim is made.

2 Definitions and conventions

Let P be a semiorder on distinct elements $x_0, \dots, x_{n-1}$. Fix a numerical representation by utilities $u_i \in \mathbb{R}$ and a threshold $\theta > 0$, with the inclusive convention

$$x_i \leq_P x_j \iff u_i \leq u_j - \theta \qquad (i \neq j). \tag{1}$$

Sort the elements by $(u_i, \text{fixed label})$, and relabel so that $u_0 \leq \cdots \leq u_{n-1}$. The fixed label breaks notation ties only; it does not add a comparability.

Give each element a coefficient vector $v_i = (a_i, b_i) \in \mathbb{Q}^2$. For an order ideal I, define

$$\sigma(I) = (A(I), B(I)) = \sum_{i \in I} v_i, \qquad L_\lambda(\sigma(I)) = \lambda A(I) + B(I).$$

Let

$$X(P, v) = \{\sigma(I) : I \in J(P)\}, \qquad Q(P, v) = \operatorname{conv} X(P, v),$$

with equal projected points deduplicated. Write $h(P, v)$ for the number of strict upper-hull vertices after retaining only the greatest B at each A and deleting points in the relative interior of an upper edge. Write $f(P, v) = |\operatorname{vert} Q(P, v)|$ for the number of strict vertices of the full projection polygon. A point has one vertex and a nonzero segment has two.

For each sorted index j, set

$$F_j = \{i < j : u_i \leq u_j - \theta\}, \qquad W_j = \{i < j : u_i > u_j - \theta\}, \qquad q_j = |W_j|. \tag{2}$$

Let $\operatorname{inc}(P)$ denote the number of unordered incomparable pairs.

3 Main bounds

Theorem 1 (Largest-index zonotope bounds)**.** *For every represented n-element semiorder P and every coefficient system $v \in (\mathbb{Q}^2)^n$,*

$$h(P, v) \leq 1 + \sum_{j=0}^{n-1} (q_j + 1) = 1 + n + \operatorname{inc}(P), \tag{U}$$

$$f(P, v) \leq 1 + \sum_{j=0}^{n-1} \max\{1, 2q_j\} \leq 1 + n + 2 \operatorname{inc}(P). \tag{F}$$

For $n = 0$, both hulls consist of the empty-ideal point and have one vertex.

Corollary 2 (Width consequences)**.** *Assume $n \ge 1$, the width of P is at most w, and $r = \min\{n, w\}$. Then*

$$\operatorname{inc}(P) \le (r-1)n - \binom{r}{2}, \qquad h(P,v) \le 1 + rn - \binom{r}{2}. \tag{3}$$

For the full polygon,

$$f(P,v) \le \begin{cases} n+1, & r = 1, \\ 2 + (r-1)(2n-r), & 2 \le r \le n. \end{cases} \tag{4}$$

The second bound in (4) is the exact maximum of the local sum in (F) under the coordinate restrictions established below. It is not a claim that all local vertices survive globally, or that the final polygon bound is attained.

4 Largest-index proof

Lemma 3 (Exact fibers)**.** *The nonempty ideals whose largest selected sorted index is j are exactly*

$$I(j,T) = F_j \cup \{j\} \cup T, \qquad T \subseteq W_j. \tag{5}$$

Proof. If j belongs to an ideal, every $i \in F_j$ satisfies $x_i \le_P x_j$ and is forced into the ideal. Largest-index maximality excludes all indices greater than j, leaving only W_j undecided.

Conversely, fix $T \subseteq W_j$. If x_k is a predecessor of a selected x_i with $i \in W_j \cup \{j\}$, then $u_k \le u_i - \theta \le u_j - \theta$, so $k \in F_j$. A predecessor of an element already in F_j is again in F_j. Therefore (5) is a lower set. The two constructions are inverse. □

Put $t_j = v_j + \sum_{i \in F_j} v_i$. Lemma 3 identifies the projected fiber as

$$S_j = \left\{ t_j + \sum_{i \in T} v_i : T \subseteq W_j \right\}.$$

Convexifying independent binary choices gives the translated planar zonotope

$$K_j = \operatorname{conv} S_j = t_j + \sum_{i \in W_j} [0, v_i]. \tag{6}$$

The fibers partition the nonempty ideals, so

$$Q(P,v) = \operatorname{conv}\left(\{(0,0)\} \cup K_0 \cup \cdots \cup K_{n-1}\right). \tag{7}$$

Lemma 4 (Planar zonotope counts)**.** *A translate of a planar zonotope with q listed generators has at most $q+1$ strict upper vertices and at most $\max\{1, 2q\}$ strict full vertices.*

Proof. Translation does not affect either count. Write $Z = \sum_{i=1}^{q} [0, v_i]$, where $v_i = (a_i, b_i)$. In upper direction $(\lambda, 1)$ its support function is

$$\max_{z \in Z} (\lambda z_A + z_B) = \sum_{i=1}^{q} \max\{0, \lambda a_i + b_i\}. \tag{8}$$

If $a_i \ne 0$, generator i changes its selected endpoint only at $-b_i/a_i$; a vertical or zero generator creates no parameter-dependent switch. At a common switch λ_0, each switching vector is a scalar multiple of $(1, -\lambda_0)$. As λ increases, a positive-a_i vector is added and a negative-a_i vector is removed, so the total movement is $\sum_i |a_i|(1, -\lambda_0)$. Simultaneous switches therefore create one exposed segment rather than cancellations or extra strict vertices. At most q switch values separate at most $q+1$ upper support vertices.

For the full polygon, every nonzero generator supplies at most one unoriented edge direction, hence at most two directed boundary edges. Repeated, parallel, antiparallel, and zero generators only reduce the count. The all-zero case is one point and the one-dimensional nonzero case is a segment, which yields the uniform bound $\max\{1, 2q\}$. $\square$

Lemma 5 (Global inheritance)**.** *Every vertex of the convex hull in* (7) *is either the origin or a vertex of a constituent K_j. Every strict global upper vertex, other than the separately counted origin, is a strict upper vertex of at least one constituent containing it.*

Proof. A vertex of a finite polytope is the unique maximizer of some linear functional. Maximizing over a convex hull of a union is the same as maximizing over that union. A global unique maximizer lying in K_j is therefore a unique maximizer over K_j and hence a constituent vertex.

For a strict upper vertex, choose a finite upper functional $L_\lambda(A, B) = \lambda A + B$ in the interior of its normal cone. This works for internal vertices. For either upper endpoint choose a sufficiently large finite value of the appropriate sign; among points with the same extreme A-coordinate, the B term selects the upper point. A vertical polygon is handled by the B functional and a singleton is immediate. Restricting the same functional to a constituent proves upper inheritance. $\square$

Lemmas 4 and 5 applied to (7) give

$$h(P, v) \le 1 + \sum_j (q_j + 1), \qquad f(P, v) \le 1 + \sum_j \max\{1, 2q_j\}. \tag{9}$$

Lemma 6 (Incomparable-pair charge)**.** $\sum_{j=0}^{n-1} q_j = \mathrm{inc}(P)$.

Proof. For an unordered pair with smaller sorted index first, say $i < j$, the reverse comparison $x_j \le_P x_i$ is impossible because $u_i \le u_j$ and $\theta > 0$. The forward comparison holds precisely when $u_i \le u_j - \theta$. Hence the pair is incomparable exactly when $i \in W_j$. Every incomparable pair is charged once to its larger index. $\square$

Substitution into (9), together with $\max\{1, 2q\} \le 1 + 2q$, proves Theorem 1.

Lemma 7 (Width summation)**.** *If P has width at most w and $r = \min\{n, w\}$, then $q_j \le \min\{j, r-1\}$.*

Proof. The elements indexed by $W_j \cup \{j\}$ have utilities in $(u_j - \theta, u_j]$, an interval of diameter strictly less than θ. They are pairwise incomparable. Thus $q_j + 1 \le r$, while $q_j \le j$ by definition. $\square$

For $1 \le r \le n$,

$$\sum_{j=0}^{n-1} \min\{j, r-1\} = (r-1)n - \binom{r}{2}. \tag{10}$$

Lemmas 6 and 7 prove the incomparable-pair and upper bounds in Corollary 2. If $r = 1$, every $q_j = 0$ and the exact local full bound is $n + 1$. For $r \ge 2$, with $g(q) = \max\{1, 2q\}$,

$$f(P, v) \le 1 + g(0) + \sum_{j=1}^{n-1} 2\min\{j, r-1\} = 2 + (r-1)(2n - r),$$

which completes the proof of the corollary.

5 Sharp relaxation and mixed-sign lower control

The coordinate relaxation used above cannot be decreased. For $1 \le r \le n$, take $\theta = 1$ and $u_j = j/r$. Then $q_j = \min\{j, r-1\}$, the semiorder has width r, and its incomparability graph is the $(r-1)$st power of a path. This realizes equality in the window schedule and in (10). It does not assert equality in the global geometric bounds.

For the order in n, consider a chain C_n with prefix ideals $I_k = \{1, \dots, k\}$. Set

$$A_0 = 0, \qquad A_k = \begin{cases} (k+1)/2, & k \text{ odd}, \\ -k/2, & k \text{ even}, \end{cases} \qquad B_k = -A_k^2,$$

and assign element k the difference vector

$$v_k = (A_k - A_{k-1}, B_k - B_{k-1}) = \begin{cases} (k, -k), & k \text{ odd}, \\ (-k, 0), & k \text{ even}. \end{cases} \tag{11}$$

The prefix projections are all consecutive integer points $(A, -A^2)$ from $-\lfloor n/2 \rfloor$ through $\lceil n/2 \rceil$, in a different prefix order. The strict concavity of $-A^2$ makes all $n+1$ points strict upper vertices and full-polygon vertices. For $n \ge 2$, (11) uses both positive and negative first coordinates. A chain is a width-one semiorder, so the upper and lower results give linear order in n for every fixed width.

6 Comparison with checked prior work

Table 1 separates the theorem from its two closest checked baselines. Eppstein's theorem applies to every represented semiorder and is stronger when width is large. Wood's result counts all ideals through cliques in the incomparability graph; it decisively subsumes the abandoned counting route but does not count supported vertices of a two-dimensional image.

Table 1: Scope of the checked comparison results.

Source	Bound or object	Exact relation to this note
Eppstein [1]	$O(n \log n)$ vertices for the full semiorder projection polygon	Unrestricted semiorder baseline; combines with the present $O(nr)$ bound.
Wood [2]	At most $2^d(n-d+1)$ cliques in a d-degenerate graph	Gives a known ideal-count bound after taking the incomparability graph; not a planar supported-vertex bound.
This note	$1 + n + \mathrm{inc}(P)$ upper vertices and an exact local full-polygon sum	Instance-sensitive and $O(nr)$ at semiorder width r; no unrestricted general-poset conclusion.

For a width-r semiorder, ideals correspond to antichains and hence to cliques in its $(r-1)$-degenerate incomparability graph. Wood's theorem gives the known bound

$$|J(P)| \le 2^{r-1}(n - r + 2).$$

The upper bound (3) equals this count at $r = 1, 2$ and is strictly smaller for $r \ge 3$. The full bound (4) equals it at $r = 1, 2, 3$ and is strictly smaller for $r \ge 4$. This numerical separation is why the planar geometry survives the prior-art collision affecting the counting proof.

7 Exact verification

A separate standard-library Python verifier reconstructs the semiorder relation, all ideals, largest-index fibers, projected points, strict upper and full hulls, width, incomparable pairs, and every displayed inequality using integers and exact rational arithmetic. It imports neither of the originating optimization engines nor a theorem implementation.

The frozen audit covers 318,667 exact cases, enumerating 3,978,057 ideals and 2,311,018 distinct projected points. Its fixtures include threshold equality, tied utilities, empty and singleton orders, zero, vertical, horizontal, parallel, antiparallel, and repeated generators, simultaneous support roots, projection collisions, one-dimensional hulls, and mixed signs. A fresh clean reconstruction in August 2026 reproduced all counts and the canonical result digest exactly. The supplementary appendix specifies the audit scope, replay command, hashes, and expected output.

These computations are finite falsification and regression evidence. They do not promote themselves to an infinite theorem and are not used as a premise in the proof above. All existing checks are internal; no external specialist review or proof-assistant formalization is claimed.

8 Limitations and non-claims

The theorem depends on the supplied semiorder representation and the inclusive threshold convention in (1). The proof allows arbitrary rational coefficient signs, projection collisions, and all listed geometric degeneracies. It does not establish that the displayed global bounds are attained, that their constants are optimal, that unrestricted semiorders have linear complexity, or that a general n-element poset has superlinear or polynomially bounded parametric closure complexity. It also does not establish historical priority. The bounded source audit supports only the statement that no exact collision was located in the checked corpus.

Data and code availability

An exact reproducibility package containing the standard-library verifier, expected output, four package tests, manifests, hashes, and a zero-mutation replay wrapper accompanies this preprint in its public GitHub repository. No restricted primary-source PDF is redistributed, and no public reuse license has been granted for the repository contents.

Automated assistance and verification disclosure

OpenAI model-assisted systems participated in proof development, code generation, adversarial testing, source organization, and manuscript preparation. Automated systems are not authors or external reviewers. The originating proof was not allowed to certify itself; separate internal implementations and reconstruction lanes attacked the stated theorem. Before submission, the named human author must inspect the complete package, approve the final text, and accept responsibility for its accuracy and integrity.

Statements and declarations

The author's affiliation or independent-researcher designation, correspondence email, funding statement, competing-interest statement, authorship approval, and license choice require final human confirmation and must be entered before submission. No value has been inferred for those personal or legal fields.

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