Supplementary Appendix to *Explicit Counterexample Families to Five Finite-Group Inequalities from Graffiti*[3]

Zackary Loevseth

1 Exact shifted-polynomial certificates

Let

$$K(q) = 1 + 21(q-1) + 140(q-1)^2 + 385(q-1)^3 + 490(q-1)^4 + 301(q-1)^5 + 84(q-1)^6 + 8(q-1)^7$$

and set $s = q - 2$. The three strict differences used in the main paper are as follows.

For C22, using $\exp(U_7(q)) \le q^3$,

$$\begin{aligned} q^{15} - q^3 + 1 - K(q) = {} & s^{15} + 30s^{14} + 420s^{13} + 3640s^{12} + 21840s^{11} + 96096s^{10} \\ & + 320320s^9 + 823680s^8 + 1647352s^7 + 2562420s^6 \\ & + 3074099s^5 + 2791985s^4 + 1856364s^3 \\ & + 851481s^2 + 240267s + 31331. \end{aligned}$$

For C21,

$$\begin{aligned} q^{15} - q^6 + 1 - K(q) = {} & s^{15} + 30s^{14} + 420s^{13} + 3640s^{12} + 21840s^{11} + 96096s^{10} \\ & + 320320s^9 + 823680s^8 + 1647352s^7 + 2562419s^6 \\ & + 3074087s^5 + 2791925s^4 + 1856205s^3 \\ & + 851247s^2 + 240087s + 31275. \end{aligned}$$

For C20,

$$\begin{aligned} q + 3q^{11} - 4K(q) = {} & 3s^{11} + 66s^{10} + 660s^9 + 3960s^8 + 15808s^7 \\ & + 43792s^6 + 84812s^5 + 112580s^4 + 97460s^3 \\ & + 49788s^2 + 11869s + 426. \end{aligned}$$

Every displayed coefficient is a positive integer. The dependency-free checker constructs these polynomials by integer-list addition and multiplication; it does not use a symbolic-algebra package.

2 Small explicit witnesses

Only the invariants used by each relevant inequality are load-bearing. The dihedral exponent values in the table are included for completeness and are not used in the C27 or C28 proofs.

Table 1: Exact invariants of the three smallest displayed witnesses.

G	$\lvert G\rvert$	$k(G)$	$\lvert Z(G)\rvert$	$\lvert G'\rvert$	$\lvert G/G'\rvert$	$\exp(G)$	Refutes
$U_7(2)$	2097152	1430	2	32768	64	8	C20–C22
D_{64}	64	19	2	16	4	32	C27
D_{50}	50	14	1	25	2	50	C28

For $U_7(2)$ the three failed comparisons are

$$1430 < \frac{2 + 3\sqrt{2 \cdot 2^{21}}}{4} = 1536.5,$$

$$1430 < |32768 - 64| + 1 = 32705,$$

and

$$1430 < |8 - 32768| + 1 = 32761.$$

For D_{64}, C27's proposed right-hand side is

$$\frac{7}{16}|4 - 64| - 4 = \frac{89}{4} > 19.$$

For D_{50}, the exact fifth-power comparison is

$$\left(\frac{14}{50}\right)^5 = \left(\frac{7}{25}\right)^5 > \frac{1}{25^2},$$

because $7^5 = 16807 > 15625 = 25^3$.

3 Database-scoped census details

The combined audit covers all applicable representatives at every order from 1 through 256, but it is not a single-version production run. The historical orders 1–128 and order-256 segments were produced with GAP 4.16.0 and SmallGrp 1.5.4; the fresh orders 129–255 segment was produced with GAP 4.16.0 and SmallGrp 1.5.5. The three hash-bound input segments were:

Scope	Rows checked
Orders 1–128 (1.5.4)	3596
Orders 129–255 (1.5.5)	744 applicable
Order 256 (1.5.4)	56092

The full decompressed-file SHA-256 values are, respectively,

```
1118dbaf9d8a1d92fc851c862852f8d27d1859c9c08b9eea57dff1f8c9dc7423
794d4d2a6b8763022c3e80f8fca55746e721517e01cbee2336f7f488705abad2
f09296de67c2192113a4f2fc3a549faed7d657475b08df538a896f736ea45dd1
```

The middle segment contains all nilpotent groups (hence all groups to which C20 applies) and flags its 92 p-groups for C22. The full producer also counts nonnilpotent representatives but does not emit conjecture invariants for groups outside both target domains.

The ten order-243 C20-negative identifiers are

$$3, 4, 5, 6, 7, 8, 9, 28, 29, 30.$$

The fourteen order-256 C22-negative identifiers are

$$515, 516, 521, 522, 3378, 3379, 3380, 3381, 3678, 3679, 6665, 6666, 6667, 6668.$$

No C20 failure appears in the checked applicable representatives below order 243. No C22 failure appears below order 256. At order 256, 322 additional representatives fail C20. Thus the combined database scope contains 332 C20 failures and 14 C22 failures through order 256.

4 Presentation-level order-243 witness

The representative SmallGroup$(243, 28)$ has

$$(|G|, k(G), \exp(G), |G'|, |Z(G)|, |G/G'|) = (243, 19, 9, 27, 3, 9).$$

Its C20 exact squaring quantities are

$$4k(G) - |Z(G)| = 73 > 0,$$

and

$$(4k(G) - |Z(G)|)^2 - 9|Z(G)||G| = 73^2 - 9 \cdot 3 \cdot 243 = -1232 < 0.$$

An exported pc presentation was reloaded in a separate GAP invocation and reproduced all six invariants. The presentation and one-row certificate have SHA-256 values

`eb1095b603e108be69a3a073f7548836e0a4013291cd3fb7708f0bb3b01f722e`,
`a8dd903a690c439945c3101a66c768751ffff7d4dd045dae23c5fffc634d9976`,

respectively.

5 Verification boundaries

The following distinctions are enforced by the reproducibility package.

1. The source-statement audit checks target identity; it does not prove the conjectures or their refutations.
2. The family checker reconstructs exact arithmetic and coefficient positivity; it treats the Pak–Soffer coefficient list as a cited premise rather than independently deriving the class-number formula.
3. The database-free $U_7(2)$ producer, the GAP matrix-to-pc workflow, and the presentation reconstruction are method-distinct internal checks. None is represented as external peer review.
4. The SmallGrp census supports only its enumerated scope. The family proofs, not the finite census, establish the universal conjectures to be false.
5. A library identifier supplies reproducibility, not mathematical identity by itself; this is why a presentation-level witness is retained.

6 Expected machine-checker output

The family checker emits a canonical JSON object with status `PASS`, schema `g3-five-family-refutations-v1`, the six $U_7(2)$ invariants, positive-coefficient degrees C20:11, C21:15, C22:15, and the witness labels C27:D_64, C28:D_50.

The combined census checker emits status `PASS`, schema `g3-combined-census-audit-v1`, first C20 failure order 243, ten C20 failures at that order, first C22 failure order 256, and fourteen C22 failures through order 256.