Explicit Counterexample Families to Five Finite-Group Inequalities from Graffiti^3

Zackary Loevseth

Abstract

Five universal inequalities for the number of conjugacy classes stated in Randy Davila's *Graffiti^3* preprint admit explicit counterexample families. For every prime power q, the upper unitriangular group $U_7(q)$ violates Conjectures 20, 21, and 22. The proof combines standard formulas for the order, center, derived subgroup, abelianization, and exponent of $U_7(q)$ with Pak and Soffer's exact conjugacy-class polynomial; after the substitution $s = q - 2$, each required strict reverse inequality has positive coefficients. Two dihedral families refute Conjectures 27 and 28: D_{2n} violates Conjecture 27 whenever n is a power of two with $n \geq 32$, and violates Conjecture 28 whenever n is odd with $n \geq 25$. The smallest displayed witnesses are $U_7(2)$, D_{64}, and D_{50}. A complete, hash-bound, segment-versioned GAP SmallGrp audit through order 256 locates the first C20 failures at order 243 (ten groups) and the first C22 failures at order 256 (fourteen groups). The family proofs are independent of those database-first observations. Exact arithmetic checkers and census manifests accompany the manuscript. No claim is made about the remaining Graffiti^3 conjectures, and no external specialist review has yet occurred.

1 Introduction

Let $k(G)$ denote the number of conjugacy classes of a finite group G. Lower bounds for $k(G)$, both in general and for nilpotent groups, form a classical line of finite-group theory [9, 8, 6, 5, 1]. The present note concerns five substantially more specific proposed inequalities involving the center, derived subgroup, abelianization, exponent, and Sylow-2 subgroup of G.

Davila's version-1 *Graffiti^3* preprint [4] presents an automated theory-selection system whose outputs are explicitly provisional and intended for proof or counterexample search. Among its finite-group outputs are Conjectures 20, 21, 22, 27, and 28. We show that each is false. The counterexamples are not isolated computational objects: the standard unitriangular family $U_7(q)$ simultaneously refutes C20–C22 for every prime power q, and elementary dihedral families refute C27 and C28.

The computational component has a narrower purpose. A complete GAP SmallGrp census of the applicable groups through order 256 shows that, within the stated segment-versioned database scope, C20 first fails at order 243 and C22 first fails at order 256. These statements are explicitly database-scoped. They neither establish historical priority nor replace the family proofs.

A bounded exact-formula and citation search conducted through 18 August 2026 found no indexed proof or refutation of the five statements. Absolute novelty is not claimed, and no external specialist has yet reviewed this manuscript.

2 The five statements

Write G' for the derived subgroup, $Z(G)$ for the center, and $\exp(G)$ for the exponent. We use $|\operatorname{Syl}_2(G)|$ for the order of a Sylow 2-subgroup, with the source-specific convention $|\operatorname{Syl}_2(G)| = 0$

when $2 \nmid |G|$. The version-1 source states the following.

$$
\begin{aligned}
&k(G) \geq \frac{|Z(G)| + 3\sqrt{|Z(G)|\,|G|}}{4} && \text{for every finite nilpotent group } G, && \text{(C20)}\\
&k(G) \geq \big||G'| - |G/G'|\big| + 1 && \text{for every finite nilpotent group } G, && \text{(C21)}\\
&k(G) \geq \big|\exp(G) - |G'|\big| + 1 && \text{for every finite } p\text{-group } G, && \text{(C22)}\\
&k(G) \geq \frac{7}{16}\big||G/G'| - |\operatorname{Syl}_2(G)|\big| - 4 && \text{for every finite } p\text{-group } G, && \text{(C27)}\\
&\frac{k(G)}{|G|} \leq |G'|^{-2/5} && \text{for every finite group } G. && \text{(C28)}
\end{aligned}
$$

The numbering and quantifiers refer only to version 1 of [4].

3 A unitriangular family refuting C20–C22

For a prime power $q = p^f$, let $U_7(q)$ be the group of upper unitriangular 7×7 matrices over $\mathbb{F}_q$.

Lemma 3.1 (Unitriangular invariants)**.** *For every prime power $q = p^f$,*

$$|U_7(q)| = q^{21}, \qquad |Z(U_7(q))| = q,$$

$$|U_7(q)'| = q^{15}, \qquad |U_7(q)/U_7(q)'| = q^6,$$

and

$$\exp(U_7(q)) = p^{\lceil \log_p 7 \rceil} \leq q^3.$$

Proof. There are $\binom{7}{2} = 21$ entries above the diagonal. The commutator subgroup consists of the unitriangular matrices whose first superdiagonal is zero, leaving $21-6 = 15$ free entries; the quotient is elementary abelian of order q^6. The center consists of the matrices $I + aE_{1,7}$ with $a \in \mathbb{F}_q$. Finally, in characteristic p, $(I+X)^{p^a} = I + X^{p^a}$; the maximal nilpotency index is 7, giving the stated exponent. The bound by q^3 follows directly for every $q = p^f \geq 2$. □

Pak and Soffer give the exact polynomial [7, Appendix A, p. 27], with $t = q-1$,

$$k(U_7(q)) = 1 + 21t + 140t^2 + 385t^3 + 490t^4 + 301t^5 + 84t^6 + 8t^7. \tag{1}$$

Theorem 3.2. *For every prime power q, the group $U_7(q)$ violates C20, C21, and C22.*

Proof. Put $K(q) = k(U_7(q))$ and $s = q-2 \geq 0$.

For C22, Lemma 3.1 gives

$$\big|\exp(U_7(q)) - q^{15}\big| + 1 - K(q) \geq q^{15} - q^3 + 1 - K(q).$$

In powers of s, the right-hand side is

$$
\begin{aligned}
&s^{15} + 30s^{14} + 420s^{13} + 3640s^{12} + 21840s^{11} + 96096s^{10}\\
&\quad + 320320s^9 + 823680s^8 + 1647352s^7 + 2562420s^6 + 3074099s^5\\
&\quad + 2791985s^4 + 1856364s^3 + 851481s^2 + 240267s + 31331.
\end{aligned}
$$

Every coefficient is positive, so the proposed C22 lower bound strictly exceeds $K(q)$.

For C21, the proposed right-hand side is $q^{15} - q^6 + 1$. Its difference from $K(q)$ is

$$\begin{aligned} & s^{15} + 30s^{14} + 420s^{13} + 3640s^{12} + 21840s^{11} + 96096s^{10} \\ & \quad + 320320s^9 + 823680s^8 + 1647352s^7 + 2562419s^6 + 3074087s^5 \\ & \quad + 2791925s^4 + 1856205s^3 + 851247s^2 + 240087s + 31275, \end{aligned}$$

again strictly positive.

For C20, the source inequality is equivalent to

$$4K(q) \geq q + 3q^{11}.$$

The reverse difference expands as

$$\begin{aligned} q + 3q^{11} - 4K(q) = {} & 3s^{11} + 66s^{10} + 660s^9 + 3960s^8 + 15808s^7 \\ & + 43792s^6 + 84812s^5 + 112580s^4 + 97460s^3 \\ & + 49788s^2 + 11869s + 426, \end{aligned}$$

which is positive. Thus all three proposed inequalities fail. □

Corollary 3.3 (A single explicit witness)**.** *For $G = U_7(2)$,*

$$(|G|, k(G), |Z(G)|, |G'|, |G/G'|, \exp(G)) = (2^{21}, 1430, 2, 2^{15}, 2^6, 8).$$

Consequently C20 requires $k(G) \geq 1536.5$, C21 requires $k(G) \geq 32705$, and C22 requires $k(G) \geq 32761$, whereas $k(G) = 1430$.

Proof. Substitute $q = 2$, hence $t = 1$, in (1) and apply Lemma 3.1. □

4 Dihedral families refuting C27 and C28

We write

$$D_{2n} = \langle r, x \mid r^n = x^2 = 1,\ xrx = r^{-1} \rangle,$$

so $|D_{2n}| = 2n$.

Theorem 4.1. *If $n \geq 32$ is a power of two, then D_{2n} violates C27.*

Proof. The group is a 2-group. For even n,

$$|D_{2n}/D'_{2n}| = 4, \qquad k(D_{2n}) = \frac{n}{2} + 3,$$

and its Sylow 2-subgroup is the whole group. C27 would therefore require

$$\frac{n}{2} + 3 \geq \frac{7}{16}(2n - 4) - 4 = \frac{7n}{8} - \frac{23}{4}.$$

The reverse strict inequality holds exactly when $3n > 70$, hence for every power of two $n \geq 32$. □

For $n = 32$, the order-64 witness has $k = 19$, while C27's right-hand side is $89/4 = 22.25$.

Theorem 4.2. *If $n \geq 25$ is odd, then D_{2n} violates C28.*

Proof. For odd n,

$$|D'_{2n}| = n, \qquad k(D_{2n}) = \frac{n+3}{2}.$$

Thus C28 would require

$$\frac{n+3}{4n} \le n^{-2/5}.$$

After multiplying by $n^{2/5}$, set

$$F(n) = \frac{1}{4}\left(n^{2/5} + 3n^{-3/5}\right).$$

Since

$$F'(n) = \frac{2n-9}{20n^{8/5}} > 0 \qquad (n \ge 5),$$

it suffices to check $n = 25$. There,

$$\left(\frac{7}{25}\right)^5 > \frac{1}{25^2} \quad \iff \quad 7^5 > 25^3,$$

and $16807 > 15625$. Hence the proposed upper bound fails for every odd $n \ge 25$. □

The first member of this displayed family is D_{50}, with $k = 14$, $|G'| = 25$, and $k/|G| = 7/25 > 25^{-2/5}$.

5 Database-scoped SmallGrp census

The family theorems already refute all five statements. We additionally assembled a complete census of the applicable GAP SmallGrp representatives at every order from 1 through 256 from three hash-bound segments. The historical orders 1–128 and order-256 segments were produced with GAP 4.16.0 [10] and SmallGrp 1.5.4 [2]; the fresh orders 129–255 segment was produced with GAP 4.16.0 and SmallGrp 1.5.5 [3]. Thus the database-first statements below concern this explicit mixed-version audit, not a single SmallGrp 1.5.5 production run.

For C20, writing $z = |Z(G)|$, the proposed inequality holds exactly when

$$4k(G) - z \ge 0 \quad \text{and} \quad (4k(G) - z)^2 - 9z|G| \ge 0.$$

For C22, the independently recomputed slack was

$$k(G) - (|\exp(G) - |G'|| + 1).$$

Table 1: First failures in the stated SmallGrp scope. "First" is database- and version-scoped, not a priority claim.

Claim	First order	Number at that order	SmallGrp IDs
C20	243	10	3–9, 28–30
C22	256	14	$515, 516, 521, 522, 3378–3381,$ $3678, 3679, 6665–6668$

For example, SmallGroup$(243, 28)$ has

$$k = 19, \quad \exp(G) = 9, \quad |G'| = 27, \quad |Z(G)| = 3,$$

so C20 proposes

$$k(G) \geq \frac{3 + 3\sqrt{3 \cdot 243}}{4} = 21.$$

An exported pc presentation was reconstructed separately and gives the same invariants.

At order 256, a further 322 representatives fail C20; these are disjoint from the fourteen C22 failures. Across the database scope through order 256 there are 332 C20 failures in total: ten at order 243 and 322 at order 256.

These observations concern the stated SmallGrp versions and ranges only. They are not used in Theorems 3.2, 4.1, or 4.2.

6 Verification and reproducibility

The accompanying exact checker independently reconstructs the three coefficient expansions in Theorem 3.2 and verifies the displayed $U_7(2)$, D_{64}, and D_{50} inequalities. A database-free C implementation enumerates conjugacy classes of $U_7(2)$; a distinct GAP workflow constructs the matrix group, converts it to a pc group, and exports a presentation for reconstruction. The census producer and checker are separate programs: the checker recomputes each arithmetic comparison rather than trusting stored truth flags.

The reproducibility package records input hashes, output hashes, expected outputs, software versions, and exact replay commands. These checks provide internal mathematical and computational validation. They are not described as external review.

7 Scope and conclusion

The source presents its conjectures as provisional outputs for expert inspection. The counterexamples here perform that intended falsification step for five exact statements. They do not bear on the correctness of unrelated conjectures or on the broader value of automated conjecture generation.

The result has two layers. The mathematical layer consists of three explicit infinite-family theorems, with $U_7(2)$, D_{64}, and D_{50} as compact witnesses. The computational layer locates the first failures of C20 and C22 within a complete, versioned database scope through order 256. Keeping those layers separate makes both claims independently checkable and prevents a finite census from being mistaken for a general proof.

AI-use disclosure

Generative-AI systems, including OpenAI Codex, assisted with source discovery, proof and code drafting, computational orchestration, manuscript organization, and language editing. AI output was not treated as mathematical evidence. The named author retains responsibility for every claim, citation, computation, disclosure, and submission field. AI systems are not authors and did not provide external peer review.

Data and code availability

The exact checkers, GAP scripts, complete applicable orders-129–255 census, public-safe certificates, manifests, and replay instructions accompany this preprint in its public GitHub repository.

The historical orders-1–128 and order-256 census files are not bundled; their exact hashes and expected summaries are recorded, and replay of the combined database-first observations requires authorized copies. The source Graffiti^3 PDF, private source certificates, rejected bytes, and the private prepublication archive are not redistributed. No public reuse license has been granted for the repository contents.

References

[1] Barbara Baumeister, Attila Maróti, and Hung P. Tong-Viet. Finite groups have more conjugacy classes. *Forum Math.*, 29(2):259–275, 2017.

[2] Hans Ulrich Besche, Bettina Eick, and Eamonn O'Brien. *SmallGrp: The GAP Small Groups Library, Version 1.5.4*, 2024. GAP package.

[3] Hans Ulrich Besche, Bettina Eick, and Eamonn O'Brien. *SmallGrp: The GAP Small Groups Library, Version 1.5.5*, 2026. GAP package.

[4] Randy Davila. Graffiti^3: Compact Theory Libraries for Automated Mathematical Discovery, January 2026. Research Square preprint, version 1, doi:10.21203/rs.3.rs-8493329/v1.

[5] Andrei Jaikin-Zapirain. On the number of conjugacy classes of finite nilpotent groups. *Adv. Math.*, 227(3):1129–1143, 2011.

[6] Thomas Michael Keller. Finite groups have even more conjugacy classes. *Israel J. Math.*, 181(1):433–444, 2011.

[7] Igor Pak and Andrew Soffer. On Higman's $k(U_n(\mathbb{F}_q))$ conjecture, 2015. arXiv:1507.00411 [math.CO], Appendix A, p. 27.

[8] László Pyber. Finite groups have many conjugacy classes. *J. Lond. Math. Soc.*, 46(2):239–249, 1992.

[9] Gary Sherman. A lower bound for the number of conjugacy classes in a finite nilpotent group. *Pacific J. Math.*, 80(1):253–254, 1979.

[10] The GAP Group. *GAP – Groups, Algorithms, and Programming, Version 4.16.0*, 2026.