Supplementary Appendix to
"Excess-Degree Bounds for Minimal Counterexamples to the Erdős–Gyárfás Conjecture"

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1 Statement identity and proof obligations

Let G be a finite simple graph of minimum degree at least three containing no cycle of length 2^k for $k \geq 2$, chosen lexicographically by minimum order and then minimum size. Let A contain the degree-three vertices, B the vertices of degree at least four, $b = |B|$, and

$$s = \sum_{v \in B} (d_G(v) - 4).$$

For $0 \leq i \leq 3$, let A_i contain those $x \in A$ with exactly i neighbors in A, and write $a_i = |A_i|$. The exact central claim is

$$|A| \geq 2b + s + 4.$$

The consequences are $3|A| \geq 2|V(G)| + s + 4$ and $|E(G)| \leq 2|V(G)| - b - 2$; when $b = 1$, parity further gives $|E(G)| \leq 2|V(G)| - 4 - \lceil s/2 \rceil$. All statements are conditional on the existence of a counterexample. They do not decide the global conjecture.

The auxiliary object F used in the proof is defined as follows. Its vertex set is B. Every $x \in A_1$ has two distinct neighbors in B; record an edge between them and retain x as the label of that edge. This is a branch encoding, not literal suppression: x still has a third neighbor in A inside G.

The load-bearing proof obligations, in order, are:

1. every proper subgraph of G has minimum degree at most two;
2. every vertex has a neighbor in A, B is independent, and $a_0 = 0$;
3. F is simple and $|E(F)| = a_1$;
4. each simple r-cycle of F lifts injectively to a simple $2r$-cycle of G;
5. F is 2-degenerate and $a_1 \leq 2b - 3$ for $b \geq 2$;
6. $4b + s = 2a_1 + a_2$ and the parity of $a_1 + a_3$ is even;
7. the cases $b \geq 2$, $b = 1$, and $b = 0$ are each closed separately.

The manuscript proves every item. No two-port or finite-census claim is in this dependency list.

2 Exact equality control

The local counting mechanism cannot replace the final constant 4 by a larger universal constant without additional information. Here is an exact finite control demonstrating that limitation.

Take $B = \{0, 1, 2, 3\}$ and $A = \{4, 5, \dots, 15\}$. The edge set is

$$\begin{aligned}\{&(0,4),(0,6),(0,8),(0,9),(1,4),(1,5),(1,10),(1,11),\\ &(2,5),(2,6),(2,7),(2,12),(3,7),(3,13),(3,14),(3,15),\\ &(4,11),(5,12),(6,13),(7,14),(8,9),(8,12),(9,10),\\ &(10,14),(11,15),(13,15)\},\end{aligned}$$

where (u, v) denotes the unordered edge $\{u, v\}$. Direct degree counting gives

$$(a_0, a_1, a_2, a_3) = (0, 4, 8, 0), \qquad b = 4, \qquad s = 0,$$

and therefore $|A| = 12 = 2b + s + 4$. The four labeled edges of F are

$$01[4], \quad 12[5], \quad 02[6], \quad 23[7].$$

Thus F is a triangle with one pendant edge and is 2-degenerate.

Exact cycle enumeration finds no 4-cycle in this graph. It does find, among others, the 8-cycle

$$0, 4, 1, 5, 2, 12, 8, 9, 0$$

and the 16-cycle

$$0, 4, 1, 11, 15, 13, 3, 7, 14, 10, 9, 8, 12, 5, 2, 6, 0.$$

Accordingly this graph is not an Erdős–Gyárfás counterexample. It is only a count-sharp structural control.

3 Independent exact computation

The source-independent checker uses Python's standard library and explicit edge sets. Its bounded scope comprises:

- 343,980 admissible integer tuples, including 1,638 equality tuples;
- all 33,866 labeled simple graphs on two through six vertices, including 26,340 2-degenerate graphs and 3,393 edge-bound equalities;
- 281,197 exact branch-cycle lifts;
- the one labeled cubic graph at order four and all 70 labeled cubic graphs at order six;
- 101 exact $b = 1$ excess values, including all 50 odd-excess strict refinements, and both equality regimes for the general case;
- the 16-vertex equality control above; and
- six intentional mutations targeting parallel branches, a nondegenerate auxiliary graph, an edge inside B, reuse of a middle vertex, the $b = 1$ square obstruction, and the parity condition.

A second implementation enumerates graphs with integer edge masks and tests cycles with precomputed masks. It reproduces every aggregate shared with the primary checker. The primary checker's canonical claim-bearing result core has digest

`cdecbd8587daa494007e5e0f8acb23b`
`5f6c34e0e9f0930da1fbc4a53c5b9b3f2`.

Both runs completed under Python 3.14.6 on macOS 15.5. The exact commands, environment, output JSON files, tests, and SHA-256 manifests are included in the public paper repository. The frozen mixed-rights private archive is not redistributed, and no public reuse license is granted.

This computation tests representations, identities, bounded graph cases, and deliberate corruptions. It does not quantify over all possible graphs, does not certify the written proof, and does not establish historical priority. Its scientific role is falsification and reproducibility.

4 Hostile audit checklist

The following failure modes were explicitly tested against the proof text: hidden connectedness; simple/multigraph drift; loops or parallel edges in F; repeated branch representatives; incorrect cycle length; use of a 3-core without checking its order; a disconnected-F error in the $2b-3$ edge bound; loss of the exact cut identity; parity before integer slack is defined; applying the $b \geq 2$ argument at $b = 0$ or 1; and substituting a finite result, a two-port statement, or the global conjecture. No mathematical defect survived the audit. The wording "suppression graph" was rejected and replaced by "labeled auxiliary branch graph."

The audit is an internal independent reconstruction, not external peer review. Specialist review remains desirable before submission.

5 Prior-art and disclosure boundary

The dated search found Carr's bound $3|A| \geq 4|B|$ as the closest same-problem result and the Narins–Pokrovskiy–Szabó bound $|E| \leq 2|V| - 3$ as a prior general density boundary. No exact or equivalent version of the excess-sensitive theorem, its A_1 branch mechanism, or the b-sensitive edge consequence was located. The durable disposition is `NOVELTY_SUPPORTED_WITHIN_SCOPED_SEARCH`, not an absolute priority claim.

Model-assisted systems participated in reconstruction, adversarial review, software, exact computation, literature search, and drafting. They are not authors or external reviewers. Authorship, affiliation, funding, competing interests, correspondence data, license, and final submission approval remain personal human attestations.